THE

MATHEMATICAL GAZETTE.

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.; PROF. H. W. LLOYD-TANNER, M.A., D.Sc., F.R.S.;
PROF. E. T. WHITTAKER, M.A., F.R.S.

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THE next meeting of this Branch has been arranged for May 30th. For further information apply to the Hon. Sec., T. G. Creak, Esq., Bron Enyri, Llanberis.

THE ELEMENTARY ARITHMETIC OF THE THEORY OF NUMBERS.*

THAT every person in the realm who teaches mathematics should be a member of our Association is an ideal towards which we are converging non-uniformly, if at all. Were this ideal state of things a reality no apology would be needed for a paper dealing with topics which though educationally important are from a mathematical point of view very familiar. As it is, the feeling haunted me as I wrote that I was writing a sermon, and that the people whom it might benefit would not be in the room, were not even members of our Association. As, however, other writers of sermons must, I suppose, sometimes find themselves in a similar position, I have ventured to go on.

Why is arithmetic taught?

If a teacher of arithmetic were required to shew cause why a child who could add up a bill and count the change for a sovereign, should not abandon the study of arithmetic, he might perhaps say that he does not teach any subject, not even arithmetic, as an end in itself, but as a means towards other ends.

Arithmetic furnishes a medium or plastic material on which the beginner may commence his training for the power of

* A paper read at the Annual Meeting of the Mathematical Association, Jan. 26, 1907. The paper was illustrated by 16 diagrams shewn as lantern slides.

sustained application and ordered effort the possession of which distinguishes the educated man. Moreover, even by arithmetic some notion of the beauty of mathematical form and the power of mathematical processes may be gained, even by those who will never receive further initiation.

Great importance is rightly attached to the power of computation. The authority of De Morgan is decisive on almost any question as to the teaching of mathematics; for he was not merely an original investigator, not merely a logician and philosopher of the first rank, but a skilful and successful teacher. De Morgan encouraged what to many persons would appear mere extravagances of calculation. A familiar instance is the satisfaction with which he recorded the names of two of his pupils who worked out a root of Newton's cubic equation to 152 places of decimals.

Now I think that certain considerations, most true and valuable in themselves, which have been strenuously urged upon us in recent years, have diverted attention from the cultivation of this power of managing large numbers, and have tended in other ways to produce a one-sided development of our subject.

The psychologist has insisted on the importance of our questions being interesting to a pupil. He points out that a boy's financial unit is a penny or perhaps a shilling, and that a question on the cost of £10,000 Bank of England stock goes outside the pupil's region of actuality. The psychologist further urges that we cannot expect appreciation of the importance of accuracy if it obviously does not matter a struck match, from any reasonable point of view, whether the answer is right or wrong.

The decimalist deplores the quaint medievalism of our national units of measurement.

The physicist urges the use of four-figure logarithms, and wants results accurate within say 1%.

In my judgment all these good people are perfectly right. But if we are limited to small quantities and approximate answers and the metric system, where are we to find food for the computer?

The object of this paper is to suggest that some experimental work on the theory of numbers may be useful as a corrective in dealing with the difficulty, and for another reason as well. By such experimental work an acquaintance with the facts—which constitute the material for deductive reasoning at a later stage—will be gained. Edward Bowen once said "the study of language is the only kind of study which deliberately professes to advance in a direction exactly the reverse of every other branch of human progress. In every other fruitful inquiry we ascend from phenomena to principles. In classical study alone we attempt to teach principles first and then advance to

facts." There has most certainly been a danger that elementary mathematics might remain on that bad eminence which, it is understood, the Classical Association is taking measures to vacate.

Again, in the theory of numbers approximate answers are no answers at all. Is there any other branch of mathematics in which an answer containing 8 or 10 significant figures can possibly be correct?

A few diagrams (which were shewn with the lantern) will illustrate my meaning.

The first shews the formation of a table of squares, thus

400	160000
	801
401	160801
	803
	161604
	805
	162409

. . . . . .

Why should not a school make their own set of tables, each boy taking ten, or twenty consecutive numbers? After multiplying out a few they will like the method of differences—very likely discover it for themselves. In this connection even Δx might be welcomed. A table of squares from 1-1000 would soon be put together by the combined effort of twenty boys each taking fifty numbers.

The cooperation of a whole class towards a definite result is in itself of no mean value. The problem of civilization which every nation must solve at peril of its existence is to reconcile order with freedom, discipline with individuality. Voluntary cooperation is the best reconciliation which the wit of man has yet devised. I think we ought not to throw away any fair opportunity for it. Nor is this the only advantage of this particular bit of computation. A boy who gets four sums right out of five is a respectable member of society. Of a man 20 per cent. of whose work is all wrong, charity itself can only hope for an early opportunity of applying the kindly maxim which bids us to speak gently of the dead. The change from the standard of the boy to the standard of the man is sometimes found abrupt.

Fortunately accuracy does not consist in never making a mistake, but in guarding against the consequences of mistake. Now if checking one's work is to become a habit, the work should in the first place be worth checking. To this I have already referred. In the second place, the checks must, to begin with, be very easy to apply, or rather, very difficult not to

apply. The table of squares satisfies both these conditions. It is obviously worth making, and the checks recur automatically at every ten steps.

Copying processes are nowadays easy and cheap, and perhaps it would be found practicable to provide a copy of the table for each boy. Cooperation might thus produce a set of tables of squares, primes, quadratic forms, and so on, which would be really useful.

Let me guard against a misapprehension by saying at once that such tables would have no *absolute* value. A table has no absolute value unless its accuracy is guaranteed, either by the test of time, or by the reputation of its computer. But such a set of tables might be a useful and valued trophy of the class room.

Now what can be done with the table of squares?

Factorising a moderately large number is notoriously no easy task, but with the table the work is systematic, and for a four or five figure number reasonably simple.

For instance

$$\begin{aligned} 2573 &= 51^2 - 28 \\ &\qquad\quad 103 \\ &= 52^2 - 131 \\ &\qquad\quad 105 \\ &= 53^2 - 256 \\ &\quad \cdots \\ &\quad \cdots \\ &= 57^2 - 676 = 57^2 - 26^2 = 31 \times 83. \end{aligned}$$

The next slide shews how the table of squares of integers from 1–1000 may be used to get the square of a six-figure number; thus

$$\begin{array}{rr} & 374231 = 1000a + b. \\ a = 374 & a^2 = 139876 \\ b = 231 & b^2 = 53361 \\ \hline & a^2 + b^2 = 193237 \\ a - b = 143 & (a-b)^2 = 20449 \\ & 2ab = 172788 \end{array}$$

$$\begin{array}{r} 139876 \\ 172788 \\ \hline 53661 \end{array}$$

$$n^2 = 140048841361$$

Again, there are many well-known results as to the representation of a number as a quadratic form. Perhaps the most celebrated is the proposition that a prime of the form $4n+1$ is, in one, and only one way, the sum of two squares.

The next slide shews the process of decomposing the prime 5869 into the sum of two squares.

$$\begin{aligned} 5869 &= 76^2 + 93 \\ &\qquad\quad 151 \\ &= 75^2 + 244 \\ &\qquad\quad 149 \\ &= 74^2 + 393 \\ &\quad \cdot\ \cdot\ \cdot\ \cdot \\ &\quad \cdot\ \cdot\ \cdot\ \cdot \\ &\quad \cdot\ \cdot\ \cdot\ \cdot \\ &= 62^2 + 2025 = 62^2 + 45^2. \end{aligned}$$

It is to work of this kind that I refer as helping to give being to the idea of mathematical form, a notion which perhaps does not receive enough attention in elementary text-books.

In this connection Euler's equation

$$(a^2+b^2)(c^2+d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$$

may be exhibited as a help towards algebra.

Algebra begins as a generalisation of arithmetic. How many text-books of elementary algebra would lead an intelligent boy to suspect this? If we follow the historic course of the evolution of our subject we shall be less liable to be asked "What is the good of Algebra?" I heard recently of a suggestion that books on Mechanics should contain plenty of algebraical conundrums to which a boy might be referred who wanted to know what was the good of algebra.

The next slide serves to illustrate, by examples familiar to all, what is meant by saying that we must bring home to a boy's mind that algebra generalises arithmetic. The slide shews the algebraic representation of an odd number, of the product of odd numbers, multiplication by a table of squares

$$ab = \left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2,$$

deduction of Pythagorean triangles by putting $b = \frac{m^2 a}{n^2}$,

whence * $(m^2-n^2)^2 + (2mn)^2 = (m^2+n^2)^2$,

multiplication of quadratic forms

$$(a^2+nb^2)(c^2+nd^2) = (ac \pm nbd)^2 + n(ad \mp bc)^2$$

and some numerical illustrations.

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* Similarly from the identity

$$24abc = (a+b+c)^3 - (a+b-c)^3 - (a-b+c)^3 - (b+c-a)^3$$

we obtain four cubes whose sum is a cube by putting

$$a = pq,$$
$$b = 3p^2,$$
$$c = 3q^2, \text{ etc.}$$

[The next slide gave, as an example of Euler's equation, the 16 decompositions of

$$5\times13\times17\times29\times37=1185665$$

into the sum of two squares.]

Another familiar bit of arithmetic, Pascal's triangle,

11	2
121	4
1331	8
14641	16
. . . .	

requires little comment save for the occasion it offers for experimental research. The queer properties of these figures are almost innumerable. Some of the suggestions in this paper are untested, but here I feel on firm ground, for a child of eight, started with the first three lines, continued correctly up to the twelfth line, and was visibly delighted at finding that the sum of each line was just double that of the preceding line.

Here, too, we have the advantage of an automatic check in the recurrence of the numbers on either side of the maxima.

In another branch of the subject consider the ancient Hindoo method of finding square root.

The mean of a and b lies between a and b.

" " $\frac{1}{a}$ and $\frac{1}{b}$ " $\frac{1}{a}$ and $\frac{1}{b}$,

and so if $$a_1=\frac{a+b}{2},\quad b_1=\frac{2ab}{a+b}$$

a_1 and b_1 are more nearly equal than a and b,

and $a_1b_1=ab$.

Continuing this process, treating a_1 and b_1 as we treated a and b, the means rapidly approach towards equality, and hence towards the value $\sqrt{ab}$. It is well known that Archimedes, in finding his approximation for π by means of perimeter of a regular polygon, required the values of certain square roots. It has been conjectured that he used this method.*

In finding $\sqrt{2}$ we have successively

$$1,\ 2;\ \tfrac{3}{2},\ \tfrac{4}{3};\ \tfrac{17}{12},\ \tfrac{24}{17};\ \tfrac{577}{408},\ \tfrac{816}{577};\ \tfrac{665857}{470832},\ \tfrac{941664}{665857}.\ \ldots\ldots,$$

Some properties of recurring series to which the work gives concrete form, are also shewn; for example

$$\begin{aligned}17&=2.\ \ 3^2-1^2=\ \ 4^2+1^2=\ \ 3+2.2^2\\577&=2.\ 17^2-1^2=\ 24^2+1^2=\ 17^2+2.12^2\\665857&=2.577^2-1\ =816^2+1^2=577^2+2.408^2\end{aligned}$$

. . . . . .

*[Cantor. Geschichte, I. 301].

Also
$$3^2=2.\ 2^2+1^2$$
$$17^2=2.\ 12^2+1^2$$
$$577^2=2.408^2+1^2 \text{ etc.}$$

Returning for a moment to the Pythagorean triangles, consider the sequence

$$\ldots\ m,\ n,\ 2n+m,\ \ldots\ldots$$

each term being formed by doubling the last term and adding the last but one.

Any two consecutive numbers of the sequence determine a Pythagorean triangle, and since

$$(2n+m)^2-n^2-2n(2n+m)=2mn-(n^2-m^2)$$

we see that the sides of all these triangles have a common difference.

Thus, starting the sequence with 1, 2 we obtain

1, 2, 5, 12, 29, 70, 169 . . .

and the Pythagorean triangles are

3, 4, 5; 21, 20, 29; 119, 120, 169; 4060, 4061, 5741; etc.

The sequences 1, 4, 9, and 2, 3, 8, . . . give triangles whose sides differ by 7, such as

8, 15, 17; 72, 65, 97; 2332, 2325, 3293. . . . .

Changing the subject once more, let the student form the remainders when the squares, cubes, fourth powers, etc., of the numbers 2, 3, 4 are divided by any prime, say 11. At first he will probably work by actual multiplication and produce a table something like this

Power, -	1	2	3	4	5	6	7	8	9	10
Numbers, -	2	4	8	5	10	9	7	3	6	1
and Remainders,	3	9	5	4	1	3	9	5	4	1

etc.

A certain amount of this labour will put him into the proper frame of mind to appreciate Crelle's method.

Suppose we wish to calculate the residues, or remainders of powers of 7 when divided by 13.

1	2	3	4	5	6	7	8	9	10	11	12
7	1	8	2	9	3	10	4	11	5	12	6
7	10	5	9	11	12	6	3	8	4	2	1

The first line contains the indices 1 . . . 12.

The second line is formed by continual additions of 7, subtracting 13 whenever possible. It gives the residues of multiples of 7 to modulus 13.

The third line contains the required system of remainders for

7^1 leaves remainder 7,
7^2 or 7×7 „ 10,
7^3 leaves the same remainder as 10×7 *i.e.* 5,
7^4 „ „ 5×7 *i.e.* 9,
and so on.

Thus the figure *following* r in line 3 is the same as the figure *below* the r in line 1.

When a table such as that just shewn has been formed the existence of primitive roots becomes obvious.

The indices of the primitive roots of a prime p such as 11 with any primitive root as base are the numbers less than p and prime to it. The formation of the complete set of primitive roots and their indices for a given prime yields some interesting results. I think the value of this kind of work arises partly from a point already mentioned, namely, that it teaches the beginner to appreciate labour-saving devices, flank attacks, or in Mach's phrase, means for economising thought.

The properties of the indices of the primitive roots give further material for experimental research, and the formation of a table such as that now shewn becomes a simple affair.

Arrangements of the indices of the primitive roots of 11.

Numbers. \ *Bases.*	2	6	7	8	
$11n+2$	1	9	3	7	Indices.
6	9	1	7	3	
7	7	3	1	9	
8	3	7	9	1	

Numbers. \ *Bases.*	7	6	8	2	
2	3	3^2	3^3	3^4	Indices as powers of 3.
8	3^2	3^3	3^4	3	
6	3^3	3^4	3	3^2	
7	3^4	3	3^2	3^3	

thus $7^{3^3}-6$ is divisible by 11.

The way in which the indices may themselves in some cases be shewn as congruous to powers of a single number is curious. It is obvious that the indices in any line are 3 times, or 7 times those in some other line.

[Other slides shewed the indices of the primitive roots of 23 arranged as powers of 7 and 13.

Thus $21^{13^9}-5$ is divisible by 23.]

The grouping of the indices in this way, and the different results when we try say 5 or 19 as indices instead of 7 or 13, give interesting examples of cyclical substitutions.

It will be noticed that we have constantly encountered illustrations of Fermat's theorem, and might by an imperfect induction arrive at Fermat's theorem. Incidentally we note that the theory of numbers gives the best possible examples of the danger of imperfect induction.

Fermat's theorem is beyond the region of elementary arithmetic, but it should not be forgotten that he is said to have got on the track of it by noticing that the fifth power of any number ends with the same digit as that number. Now, to my mind, it is not reasonable to expect to gain a comprehension of Fermat's theorem more cheaply than did Fermat himself. That is to say, an acquaintance with Fermat's theorem as an experimental fact should precede any deductive proof.

The last slide relates to the factorisation of Mersenne's numbers, viz. 2^n-1 when n is prime. Fermat himself* indicated the method of predicting the factors of numbers of the type shewn in the slide, thus

$$25-23=2,$$
$$\therefore\ (5^2-23)^{11}=2^{11},$$
$$\therefore\ 5^{22}\equiv 2^{11} \text{ mod. } 23,$$
$$\therefore\ 2^{11}-1\equiv 0 \text{ mod. } 23.$$

Again
$$77^2-5927=2,$$

and hence in the same way

$$2^{2963}-1\equiv 0 \text{ mod. } 5927.$$

In fact if n and $2n+1$ are primes, and 2 is a quadratic residue of n, then 2^n-1 is divisible by $2n+1$.

In spite of the labours of some of the greatest intellects, it is hardly too much to say that no progress has been made towards *predicting* the factors of Mersenne's numbers in other cases, and thereby rediscovering the secret which it is almost certain was buried with Fermat.

Is there any other branch of mathematics in which we can shew a beginner problems which are within his comprehension and yet which are still unsolved? Can we in any better way bring home to him the fact that mathematics is not a dead but a living subject? C. S. JACKSON.

* Œuvres, ed. Tannery, II. 210.

THE TEACHING OF MATHEMATICS IN ENGLISH PUBLIC SCHOOLS FOR BOYS.*

1. In the first place, the present paper refers to conditions prevailing in England only, to the exclusion of Scotland, Ireland and Wales.

2. Secondly, the term "public school" has, in England, a rather special meaning. A public school is generally understood to be an endowed secondary school of fair size, educating boys of the upper and middle classes, independent of state control except in so far as its income is supplemented from public funds. Public schools may be either boarding schools or day schools, but most of them belong to the former category.

Other classes of secondary schools exist, such as the unendowed day schools maintained by local taxation; these are increasingly numerous and important, and in many cases compete victoriously with the weaker endowed schools of the neighbourhood. These schools differ from public schools in that their educational policy is controlled in the main by a government office, the Board of Education.

3. The public schools monopolize the education of the upper classes of this country. They have always prided themselves on their freedom from external control; each school is governed by its own governing body and headmaster, and generally the latter is allowed to have a free hand.

This apparently complete freedom is tempered by the following causes. In the first place, many public schools find it advisable to accept pecuniary help from the state, and this entails inspection by the Board of Education. The richest and most important schools, however, are able to dispense with this aid; but of late years there has been a disposition on the part of the leading schools to submit their work voluntarily to inspection.

Secondly, it must be admitted that, in spite of this independence, these schools are not really free; the instruction given is determined mainly by a variety of public examinations, such as the Scholarship examinations of the Colleges at the Universities of Oxford and Cambridge, the preliminary examination for degrees at the Universities, the school examinations for certificates of proficiency conducted by the Universities, the examinations for entrance to the Army conducted by the Civil Service Commissioners. The status of a school in the eyes of the public, and its power of attracting pupils and paying its way, depend largely on the success it can shew in these various competitions. Few reforms can be accomplished unless the examining authorities approve, *e.g.* the study of Greek is compulsory for all boys aspiring to a degree at the Universities of Oxford and Cambridge, and the public schools are in effect compelled to teach Greek to all such boys. It will be seen that the action of examining bodies has been all-powerful in the recent history of mathematical teaching in this country.

The present paper refers to the conditions prevailing in these public schools, and we will now give some account of their organization for mathematical teaching.

4. A public school is divided into "forms" or classes; the classification is determined mainly by proficiency in literary subjects such as Greek, Latin, English, History, Geography, Scripture Knowledge. These subjects are taught by the form master.

For purposes of mathematics, a group of forms constitutes a "block," and the boys in this block are redistributed into "sets" according to their proficiency in mathematics. A school of 400 may be divided into 4 blocks; and a block of 100 boys may form from 4 to 6 sets, the number of boys in

* Paper read before the International Congress of Mathematicians, held at Rome in April, 1908, by Mr. C. Godfrey, Headmaster of the R.N. College, Osborne; formerly Senior Mathematical Master at Winchester College.

a mathematical set varying from 25 to 15. The reasons for this system of re-distribution for mathematics are: (1) The "form master" does not generally possess an expert knowledge of mathematics. (2) The boys in a given form are always found to differ too widely in their mathematical ability and attainments to be taught together without re-distribution.

The number of hours given to class-teaching in mathematics is 4-7 weekly, with 1 or 2 hours of preparation out of school. Many schools have a "Modern" department, in which the place of Greek is taken by German and additional mathematics and science. A large proportion of the time in class is spent by the boys in working exercises on paper, the master going among them and giving help when it is needed.

All the different branches of mathematics are commonly taught in the same class by the same master; but in a few schools a different arrangement exists, and boys are grouped in one system of classes for Algebra and Arithmetic, and in another system for Geometry.

It may be stated here that a similar scheme of rearrangement is usual for the teaching of modern languages and science respectively.

5. During the last two years at school, between the ages of 17 and 19 there is a strong tendency for boys to specialize, the subject chosen for special study being generally one of the following: Classics, Mathematics, Science, Modern Languages, History. This early specialization is the result of the policy of the College authorities at Oxford and Cambridge, who award Scholarships for excellence in one subject; *e.g.* a boy who showed fair proficiency in both classics and mathematics would not stand a chance against another boy who was really good at one subject only.* The consequence of this is that a candidate for a classical scholarship will often drop all mathematical and scientific work at the age of 17. Similarly, a promising mathematician will often pass the last two years of his school life with little or no literary instruction, though it is only fair to say that there is a great difference between one school and another in this respect. In any case, the competition between schools for scholarship success tends to defeat the attempts of those teachers who believe that instruction should be on broad lines till the end of the school period.

6. Before trying to indicate the standard attained in the various mathematical subjects, it may be well to say a word about the considerable changes that have taken place in mathematical teaching during the recent years, changes that have overturned many landmarks and altered the standards of attainment in various directions.

7. Every subject of instruction may be viewed in two lights, according as it is appreciated for its utilitarian or for its disciplinary value. Perhaps teachers are apt to fix their attention on the latter aspect, while the general public regard the former as more important. It is not necessary to labour the point that a one-sided view is mistaken in either case. Possibly it could be maintained that almost every subject of instruction found its way into the curriculum for the sake of its practical utility; and that teachers, discovering a disciplinary way of presenting the subject, are inclined to retain it on this ground long after some change of circumstance has destroyed its practical value.

However this may be, it would seem that ten years ago mathematics was regarded in public schools mainly as a mental gymnastic. This exaggerated stress defeated its own object, as boys could hardly be induced to take an interest in a subject taught from motives that were to them foolishness. Good teachers felt the need of changes that would bring a breath of fresh air into the class-room and put more actuality into their work. But little change was possible; the existing system was stereotyped by the public

* In some cases a scholarship can be won for a combination (1) of mathematics and some physical science, (2) of classics and history.

examination papers; and it was clear to all that, though the time were ripe for change, no change would come till some wave of public opinion should carry examiners and teachers together to a new position.

8. The needful impulse came from the engineering profession. The times had gone by when engineers despised mathematics and relied on common sense and intuition combined with a large factor of safety. Nowadays they say that their young men cannot know too much mathematics, provided that it is mathematics of the right kind. The new epoch has been marked by the rise of a successful department of engineering at the University of Cambridge, a department whose graduates readily find openings on leaving the University.

The engineers, then, were ready to value mathematical teaching in schools; but they complained that the teaching actually given was not on practical lines. Matters came to a head at the Glasgow meeting of the British Association in 1902, when Professor J. Perry, Professor of Mechanics at the Royal College of Science in London, made an attack upon the existing state of things.

9. As an outcome of this movement, various committees were formed in which the views of the practical men and the schoolmasters were compared. It was found that substantial agreement was possible on most points. Schoolmasters realized that useful subjects could be made as educational as the conventional futilities that had come to be identified with school mathematics. Just as advanced research in pure mathematics gains in dignity and purpose by a close connection with the problems offered by the physicist, and on the other hand may become aimless and unreal when divorced from its applications, so school mathematics found its salvation in a renewed alliance with the countless applications suggested by modern industrial life.

The Universities and most other examining bodies consented to alter their regulations and papers so as to fall in line with the new views. A fresh situation was thus created for all teachers and students of mathematics; and it is perhaps possible at the present date to review the gains or losses arising therefrom.

10. **Arithmetic.** This subject has won its way but slowly to a respectable position in English schools. It was regarded at one time solely as a handmaid to money-reckoning and commerce. The time was spent, busily but unprofitably, in mastering the complexities of the British system of money, weights and measures. Arithmetic was not taught in its proper relation to other branches of mathematics. Financial questions took up too much time; and, as might be expected, these often became singularly unreal in the hands of the schoolmaster. A more serious evil was the host of special and tricky problems that cumbered the course. Any problem set in a public examination was pounced upon by the text-book writers and elevated into a type, with a chapter of its own, to be solved by special and ingenious methods. Thus we find in current text-books chapters on pipes filling and emptying baths, races and games of skill, cows grazing on a field of uniformly grazing grass.

The subject naturally fell into contempt, and it was observed that at the age of 18 boys were quite unable to make any useful application of arithmetic, and were even ignorant of the decimal system. What arithmetic boys had learned was useless in practical life; and depending on a collection of special tricks rather than on a few simple principles, it was equally useless as an educational training.

So undignified had the subject become that competent mathematicians neglected it at school, and were often almost helpless in the presence of figures and numerical results. Never by any chance were numerical data or illustrations introduced into Geometry. It was very unusual to meet with any demand for numerical work in the papers on advanced mathe-

matics set at the Universities. The result of this is that the knowledge of the mathematician trained under this system is almost entirely *qualitative*: it would seldom occur to him to apply any *quantitative* test to his work, unless his subsequent experience has been such as to correct the effects of early training.

11. The aim of recent reforms in arithmetic has been: (1) *Destructive*—to simplify the subject, to rid it of unnecessary special rules and devices, to omit artificial types of problems whose original interest has evaporated, to lay less stress on financial arithmetic.

(2) *Constructive*—to insist on accuracy and facility in the simplest operations with integers and decimals, to insist on perfect comprehension of decimal notation and the metric system, to bring home to the pupil what an extremely small number of separate principles and rules he has to master, and that, for the rest, he may rely on his common sense.

12. Some of the best teachers are inclined to emphasize the study of the *theory* of arithmetic, *e.g.* to lay stress on rigorous proofs of fundamental operations with vulgar fractions, to examine minutely the degree of approximation arising from a given series of calculations, etc. Others hold that, while these studies are doubtless worthy of passing mention, they are for the most part a good deal too difficult for a thorough treatment at this stage; they would prefer to postpone them till the pupil has gained more maturity and learned a fair amount of algebra.

13. The use of 4-figure logarithm tables is coming into fashion. Ten years ago the only tables found in the mathematical class-room were those of seven figures, which were used in the solution of triangles. These were not handy enough, and the boy never had enough practice to use logarithms with confidence. The science teachers, however, had discovered the practical usefulness of 4-figure tables, and complained that they had to do the work of their mathematical colleagues in teaching the use of logarithms. This is ceasing to be true; it is found that boys of 14 can be taught to use 4-figure tables willingly and with comprehension; the range of possible operations has been widened considerably thereby.

14. Under the heading of arithmetic it is convenient to refer to two cognate matters, *the importance of numerical work in all subjects*, and the introduction of *laboratory work* into mathematical teaching.

The importance of numerical work in all subjects. The excision of useless matter from the arithmetic course might conceivably have had this disadvantage, that the pupil would thereby lose opportunity of practising numerical manipulations. This danger has been avoided by the free use of numerical work in other branches, more especially in geometry and trigonometry. In every branch stress is laid on the necessity of rough numerical checks. The movement will be traced later on under the various headings; here it is enough to point out that it leads to (1) facility in numerical operations, (2) a more vivid realization of the results thus illustrated.

15. **Laboratory work in Mathematics.** Many schools now arrange that boys of 13-15 shall take, as part of their mathematics, a course of experimental work in the laboratory. During this course they are taught to measure and weigh (incidentally learning to realize the advantages of the decimal notation), to determine the surfaces and volumes of actual objects, to determine densities and specific gravities, to discover the simpler laws of hydrostatics, etc. Perhaps the actual store of knowledge gained during this course is not great, but there is little doubt that it suggests a practical and workmanlike view of mathematics, and that it satisfies the need for co-ordination between brain, eye and hand, which many teachers believe to be inherent in the nature of the British boy.

16. **Plane Geometry.** The most striking changes have taken place in the teaching of geometry. Five years ago the Universities and most other examining bodies demanded Euclid's sequence of propositions. Euclid's

actual proofs were not demanded; but no proof was accepted that violated Euclid's logical sequence.

This had long been found to be a trying restriction. It seemed probable that Euclid's sequence could be improved upon in some respects. But the most serious aspect of the case was this, that the restriction consecrated and stereotyped a lifeless style of teaching. A keen teacher found his hands tied. There was little room for originality or freshness of presentation. No doubt much sound teaching was given; but the mass of schoolmasters were content to ask for a more or less intelligent memorizing of proofs.

They regarded the solving of exercises or "deductions" as beyond the boy's powers. The constructions were very rarely performed with actual instruments. A great proportion of the boys were unfamiliar with the concepts about which they were expected to reason; *e.g.* it was not unusual to find that a boy had been through Euclid's Book II. (areas of rectangles) without distinguishing between "rectangle" and "right angle."

17. The reforming party held that a more vivid realization of the shapes and properties of geometrical figures was necessary before these properties could profitably be made the subject of strict logical treatment. The logical training obtainable from geometry they valued no less than did the conservatives. But they urged that, if the logic is to be more than mere word-play, an interested familiarity with the subject matter must come first.

Let us take as an illustration the theorem of Pythagoras about the squares on the sides and hypotenuse of a right-angled triangle.

The old view of geometry teaching was as follows:—Here is a remarkable fact. We will show the class that it is possible to start from the very simplest principles, to argue by steps that will convince the most ignorant, and finally to arrive at this astonishing result. The new school of thought, admitting the necessity and the value of the above proceedings, maintained that more was needed. It is necessary, they said, not only to astonish and conquer by the force of pure logic, but also to give power and mastery and the capacity of applying this weapon of logic to new conquests. In the case of Pythagoras' theorem, the desire for a logical proof may not arise till the pupil has been convinced in other ways that the facts are as stated. He should measure the sides and calculate the squares, he should verify the equivalence by dissecting and fitting together (and perhaps by weighing). Moreover, the facts should not be stated crudely as a subject for verification; the matter should rather be presented in such a way that the boy may have a chance of thinking ahead of the teacher and anticipating the discovery. In every way he should be encouraged to lead rather than to follow.

18. Geometry, in fact, was conceived to be a suitable subject for experiment. To experiment in geometry, a child must learn to draw and measure with sufficient accuracy. He may experiment in other ways too; *e.g.* by cutting and folding paper, by using square-ruled paper, transparent paper, string, wooden blocks, etc. But he can hardly progress far without a fair amount of skill in handling simple drawing instruments.

For this reason then, in order to experiment, he must be provided with graduated ruler, protractor (for measuring angles), compasses, and set-square (for drawing perpendiculars and parallels).

For another reason, too, these instruments must be provided. Problems of construction are unmeaning unless it is specified what instruments are allowed. The problem of trisecting a given angle is solvable if a straight edge is allowed on which a given length may be marked. But the problem is not solvable by means of the instruments allowed by Euclid. Problems of construction, then, cannot be undertaken intelligently unless the learner understands these instrumental restrictions; and he is not likely to understand the restrictions unless he actually handles and uses the legitimate instruments.

Again, geometrical instruments satisfy the child's need for bodily activity. He thinks better if he is using his fingers. Ideas are suggested to him by the action of drawing figures. His attitude becomes active instead of passive.

19. Arguments such as the above were used to justify the use of instruments in the class-room. And of course it was not forgotten that geometrical drawing has a utilitarian value in many professions, *e.g.* in engineering, architecture, navigation, and in military work. Under the old system the instruction in geometrical drawing had been divorced from the study of theoretical geometry, to the detriment of both studies. It was frequently taught as a branch of the fine arts rather than as mathematics. One result of this has been an exaggerated respect for artistic finish and "inking-in"; but the worse feature was that "geometrical drawing" came to be identified with a vast collection of special and unrelated rules; the educational value of the subject had sunk to zero.

20. The teachers' demand for a freer and more experimental system of geometry teaching sprang from the belief that a more intimate knowledge of geometry would increase its value as a logic-training subject. On the other hand, the engineers and other outside critics cared little for logic training, but greatly desired that their students should have some knowledge of geometrical facts, which was admittedly not given by the existing system.

21. However different the ultimate objects in view, the demand for definite change was too urgent and unanimous to be resisted. The universities remodelled their schedules of examination. The University of Cambridge set the tone of the reform by (1) requiring the use of instruments, (2) accepting any proof of a theorem which should "appear to the Examiners to form part of a systematic treatment." It published a modest list of theorems and constructions which were to be considered fundamental. The list omitted many of the less useful and interesting propositions of Euclid's treatise. The second book of Euclid (areas of rectangles) was recognized to be unsuitable for formal logical treatment; its chief propositions were introduced as a "geometrical illustration of algebraic identities."

An important step was taken in admitting "proofs which are only applicable to commensurable magnitudes." Euclid's treatment of proportion is strict and covers all magnitudes, commensurable or incommensurable. It is of the greatest interest to a mature mathematician, and might suitably form part of a university course; though for this purpose a more modern treatment of incommensurables might possibly be preferred. But Euclid's theory was a stumbling block to beginners; and indeed, as generally taught in English schools, it was unintelligible, since Book V. was always omitted. The university decided that the theory of similar figures might be studied in schools without attacking prematurely the much more difficult theory of incommensurables.

Hypothetical constructions were allowed; *e.g.* in proving the equality of the base angles of an isosceles triangle, it was considered legitimate to use the bisector of the vertical angle even though a ruler and compass construction for this bisector might not have been given and proved at this stage. An existence theorem is, in fact, assumed; viz. that an angle has a bisector. The choice of any particular method for drawing the bisector is not considered relevant to the discussion.

22. The change of regulations released a volume of pent-up energy. Every enthusiast felt that he might now teach geometry in his own way. Numerous text-books were produced, embodying every shade of opinion. As was to be expected, many of the new developments were extravagant and had no permanent effect. In particular, there was temporarily a tendency to over-emphasize the practical and experimental side of the

work. But it was soon realized that this would make the subject invertebrate, that there must be a certain element of severity in every school study, and that for purposes of general education geometry must still stand or fall by the logical training it gives. The most widely used of the new text-books did not differ from Euclid in their method of setting out and combining theorems. As for the experimental work, some books confined this to an introduction, while others preferred to allow experiment and theory to develop side by side. In all cases, constructions were expected to be done practically with instruments; and numerical data and illustrations were used freely. As regards sequence of theorems, no system has found favour that can be called revolutionary; there has been no bold cutting adrift from the Euclidean tradition. Among the variations from Euclid adopted by different authors are the following: (1) A rearrangement of the earlier theorems into the order:—angles at a point, parallel lines, angles of triangles and polygons, congruent triangles. (2) The presentation of the theorems relating to areas of triangles, parallelograms and polygons, rather in the light of mensuration rules than of geometrical theorems.

23. As to the effects of all these changes on education, perhaps it is still too soon to give a final opinion. The transitional period is still on us. No doubt boys find geometry a much more interesting subject than they used to. They have more practice in solving exercises for themselves; they no longer find this a hopeless undertaking. They are able to *see* more in a geometrical figure. They have gained a sense of power, and can easily be induced to take up little bits of research work for themselves, *e.g.* in plotting loci or in devising mechanical models to illustrate various points.

On the other hand, much less time can be given to writing out formal propositions; and there is some reason to think that boys have lost something of their power of expressing geometrical reasoning in words. Doubtless this is a fault that will correct itself in course of time; and perhaps it is as well that the stereotyped Euclidean language should give place to a more individual wording, even at the cost of some initial loss in accuracy. At one stage of the movement there seemed to be a danger that experimental verifications would be taken for proofs; but the distinction has been emphasized so repeatedly that perhaps this particular reproach can no longer be aimed at the new methods.

24. **Geometry in three dimensions.** The position of this subject in schools is still unsatisfactory. It does not form part of the geometry demanded for the preliminary examinations of Oxford and Cambridge, nor is it suggested here that it could with advantage be included in these schedules.

When a boy has covered the elementary course of plane geometry, he has presumably received an adequate training in logical method, in so far as this can be given by geometrical studies. If he goes on to the geometry of three dimensions (Solid Geometry) the first object should be to gain the power of realizing mentally the relations of figures in space; he must learn to "think in space." Few of the boys who used to study Euclid's Book XI. gained this power; and for this reason one can hardly regret that nowadays this book is hardly read at all in schools.

For many years an attempt was made by the Science and Art Department (now merged in the Board of Education) to encourage the study of solid geometry. Public examinations were held in the so-called "Descriptive Geometry,"* *i.e.* the representation of solids by means of plan and

* This is the "Geometrie Descriptive" of Monge, and must not be confounded with the descriptive or non-metrical geometry of Chasles and other writers.

elevation, and by perspective projections. The same body used to examine in plane geometrical drawing; and, unfortunately, the examination system succeeded in reducing both studies to mere collections of special tricks, often taught by teachers who had received no mathematical training. The attempt ended in failure; and the effect on public schools was nil, as these schools made no use of the department's examinations.

25. A satisfactory course of instruction in three-dimensional geometry should include:

(1) The determination of the surfaces and volumes of the elementary solids.

(2) A discussion of the relations of points, lines and planes in space. This should be comparatively informal, and freely illustrated by all kinds of devices, such as string and cardboard models, stereoscopic views, dissected solids, etc. Much of it might be worked in incidentally during the plane geometry course; *e.g.* when parallel and perpendicular lines *in plano* are before the class, it is very profitable to discuss parallel and perpendicular lines and planes in space. Indeed, should not the very earliest geometry teaching begin by dealing with concrete solids, and then pass on to abstractions such as points and lines?

(3) A course of the really fundamental constructions in descriptive geometry. Probably this course if taught intelligently would do more than any other to cultivate the power of thinking in space.

No accepted or standard course exists which embodies these features. University-trained mathematical teachers are as a rule quite ignorant of descriptive geometry. This reproach will gradually disappear, as descriptive geometry is required in the new honours course at Cambridge: in the meantime, it is to be hoped that this subject will gradually win its way in public schools.

26. **Algebra.** The reform of geometry teaching was accompanied by some activity on the side of algebra. Many teachers and examiners held that the teaching had laid too great stress on manipulative skill, at the expense of intelligent study of the why and wherefore. Certainly much time was spent in working out long graduated sets of exercises on factors, equations, fractions, etc. There was a kind of rebellion against this practice, and teachers sought to lighten the "toil" by introducing graphs, logarithms, tables and other interesting matters at a comparatively early stage.

All this had a decidedly stimulating effect. It is a revelation to a boy to learn that a function of a variable may be associated with a curve; that he can solve equations, extract roots, etc., by graphical methods.

Following the usual law, the reform went too far. Some teachers and text-books were not content to touch lightly on graphs, immensely suggestive as this is for a boy of 13. They enlarged the subject until it amounted to a premature attack on analytical geometry. There was a certain tendency to abandon analytical in favour of geometrical and graphical methods. The laborious working of algebra exercises was cut down more and more, until there was a danger that the boys would become helpless in dealing with the most straightforward algebraic expressions.

The pendulum is now swinging in the opposite direction. If its oscillations can be damped in time, we may hope to settle down to a system which shall give reasonable skill in quite straightforward manipulation, and at the same time set graphical work in the true position it should occupy in elementary teaching.

27. To speak frankly, it is not easy to determine the exact function of algebra teaching in a secondary education. It may be admitted that the conception of algebra as a generalized arithmetic is of great educational value. If a boy can be made to see that a single algebraic formula is a kind of portmanteau, into which are packed an infinite number of arithmetical data—if this can be brought home to him, he has gained one of the most fruitful ideas that his mathematical education can give

This alone would justify the teaching of algebra, and this end can be attained without giving any great amount of time to exercises in manipulation.

The above is an example of what may be called the *ideas* of algebra. Everyone is the better for acquiring these ideas. The majority of public school boys will never have occasion in their after life to make any direct use of the mathematics they learn at school: for this large class one is tempted to think that the *ideas* should suffice, together with the minimum of manipulative practice that will suffice to make these ideas intelligible.

For those, on the other hand, who will have to *use* mathematics in after life, the case is very different. For them algebra is an indispensable tool, a key to unlock the armoury. Without facility in handling algebraic expressions their path will indeed be thorny. They cannot shirk the drudgery of exercise-working; and for the mathematician, as for the musician, skill comes only as the reward of long and strenuous practice, practice which is perhaps of slight value in itself, and only desirable for the sake of the end in view.

The above distinction between the amateur and the professional student of algebra is perhaps of small practical value in England; the fact being that, in the main, the object of English education is the passing of examinations. Examinations can test skill, but not ideas; hence it follows that all English boys tend to be taught as if they were destined actually to make use of mathematics in their after life.

28. **Plane Trigonometry.** This subject is now introduced at an early stage in many schools, say at 13-15 years of age. It is held that every school boy ought to be able to learn some trigonometry as an extension of his geometry.

This early introduction has become possible through the practice of laying increased stress on what may be called *numerical trigonometry*. An introductory course of numerical trigonometry deals with the sine, cosine, tangent of acute angles; the graphical determination of these functions; the use of tables; the solution of right-angled triangles, and of practical problems depending on them; this work being closely related to the drawing to scale which now forms part of geometry teaching. Other triangles are solved by splitting them into right-angled triangles.

Treated in this way, the beginnings of trigonometry present no difficulties. The feasibility of the work at this stage seems to depend on

(1) A very gradual advance, and presentation of difficulties one at a time; *e.g.* to use logarithmic sines at this stage would lead to confusion.

(2) The postponement of what may be called *algebraic* trigonometry, *i.e.* the manipulation of expressions containing trigonometrical functions.

The solution of practical problems is often based on observations made by the pupils with a simplified form of theodolite.

When the elements have been thoroughly digested, it is found that progress on the ordinary lines is fairly rapid.

29. **Mechanics,** including *Statics* and *Kinetics*. There is no uniform practice as to which of these subjects is to be taken first. The graphical movement of late years, however, has tended to set statics in the earlier place.

The best usage nowadays bases statics upon an experimental course in the laboratory, or mathematical workshop. Here the pupil establishes the parallelogram of forces; the laws of moments; of friction, etc.; and learns about the principle of work by experiments on a variety of simple machines. Subsequently or simultaneously he goes through a course of graphical statics; which is gradually combined with the analytical treatment for which the pupil's work in trigonometry has been preparing him. All this can be done at the age of 15-16.

If this course is adopted, kinetics come later. It is not easy to arrange experimental work in this subject; and the teaching is for the most part merely theoretical. The average boy seems to find kinetics much more difficult than statics; and perhaps trigonometry and statics will for some time form the limit of the mathematical studies of most boys in public schools.

30. **Advanced School Mathematics.** Boys who intend to take up mathematical studies at the universities are expected to make some headway in the following subjects: Modern geometry, analytical geometry, the geometry of the conic sections treated both geometrically and analytically, higher algebra, trigonometry, and mechanics, differential and integral calculus.

Modern Geometry includes the geometry of the triangle, the properties of pole and polar, and of coaxal circles, inversion, reciprocation, orthogonal and conical projection, etc.

Geometrical Conics is a subject to which great and perhaps exaggerated importance has been attached in English schools. This may be due to the fact that Newton was compelled to throw his principia into a geometrical form, his contemporaries being unable to appreciate the method of fluxions by which his results were attained.

Analytical Geometry, mainly of the straight line, circle and conic sections. Here again the conic sections bulk very large; they are studied in great detail, and the finished schoolboy mathematician is expected to attain considerable skill in the application of such methods as trilinear coordinates. The modern tendency is to try to reduce the treatment (both analytical and geometrical) of the conic sections to its true proportion; and to spend longer time on the more fruitful methods of the calculus.

Higher Algebra. A miscellaneous and unscientific category, covering summation of series, convergency, continued fractions, theory of numbers, inequalities, probability, theory of equations, etc. This catalogue of subjects might be startling if it were not explained that only elementary and isolated propositions are studied.

Higher Trigonometry. A species of algebra usually classified by schoolmasters as a separate subject. Complex numbers first make their appearance at the introduction of this subject. This earlier work boys find very attractive; later on the treatment of infinite expansions and product tends to become laborious, and many teachers think that the pressure of examinations over-emphasizes this section of the work.

Differential and Integral Calculus used to be regarded as the coping-stone of school mathematics. But there is a strong movement at the present day in favour of an early use of the calculus. Perhaps it is hardly ascertained as yet how early this may be attempted; but it has been proved that a slight knowledge of differentiation and integration simplifies and generalizes the study of analytical geometry and kinetics; subjects to which tradition assigns an earlier place.

C. GODFREY.

R.N. College, Osborne.

ON BINAL FRACTIONS.

BY LT.-COL. ALLAN CUNNINGHAM, R.E., FELL. OF KING'S COLL. LOND.

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1. *Binal Fractions.* The reciprocals $(1/N)$ of any number (N) will be termed *Binal Fractions*, or shortly *Binals*, when expressed in the binary scale. The notation used is similar to that of ordinary *decimals.* Thus (see table at end):

1°. A point is used to mark the beginning of the fraction.
2°. The binal ($1/N$) consists of a certain sequence of ciphers and units (the only digits used) to the right of the point; partly non-recurrent and partly recurrent.
3°. The beginning and end of the recurrent portion (or repetend) are marked by a point placed over the first and last figure of the repetend.

2. *N a power of* 2. Let $N=2^a$; then

$$\frac{1}{N}=\cdot 000\ldots 01,\ [(a-1)\text{ ciphers, one unit}]. \quad \ldots\ldots(1)$$

Note that there is *no recurrence*. $\ldots\ldots(1a)$

2a. *N odd.* Let $N=\Omega$ (an *odd* number).

Then $1/N$ consists *solely of a repeating cycle*. $\ldots\ldots(2)$

2b. *N even.* Let $N=2^a.A$, where A is *odd*; and let P be the period of $1/A$ (which is wholly *repetend*). Then

$1/N$ consists of a group of a ciphers, following the binal point, *non-recurrent*, due to the factor 2^a, followed by the repetend period (P) of $1/A$. $\ldots\ldots(3)$

Hence, it is unnecessary to consider the reciprocals of *even* numbers any further, as they can be at once written down by the above Rule (3), when the repetend cycle of $1/A$ is known.

3. *Number of figures in period.*
Let N be an *odd* number $>2^a$, but $<2^{a+1}$.
„ P be the binal repetend cycle of $1/N$.
„ n=number of digits in P.
„ C=number of ciphers in P.
„ U=number of units in P.
„ c=number of leading ciphers in P (*i.e.* following the binal point).
„ ξ be the Haupt-Exponent of 2 (mod N) [*i.e.* the *least* exponent giving $2^\xi\equiv+1\pmod N$].

Then
$$C+U=n=\xi. \quad \ldots\ldots(4)$$
$$c=a. \quad \ldots\ldots(5)$$
$$U\geqslant C,\ U\geqslant\tfrac{1}{2}\xi,\ C\leqslant\tfrac{1}{2}\xi. \quad \ldots\ldots(6)$$

Thus, in all cases, the leading a figures (*i.e.* next to the binal point) are ciphers, *followed by a unit*, and the period of ξ figures must *end with a unit*. In what follows, ξ is always supposed known;* so that only $(\xi-a-2)$ figures of P are left to be determined.

4. *Binal expression of* $1/N$. By definition, P is one period of the repetend of $1/N$, expressed as an integer, and contains ξ figures. Hence $P/2$ is the real value of the first period of $1/N$, and therefore

$$\frac{1}{N}=\frac{P}{2^\xi}+\frac{P}{2^{2\xi}}+\frac{P}{2^{3\xi}}+\text{etc. } ad\ inf. \quad \ldots\ldots(7)$$

$$\therefore\ 2^\xi.\frac{1}{N}=P+\left(\frac{P}{2^\xi}+\frac{P}{2^{2\xi}}+\frac{P}{2^{3\xi}}+\ldots ad\ inf.\right)$$

$$=P+\frac{1}{N}$$

*In the author's Paper on *Haupt-Exponents of* 2, in the *Quarterly Journal of Pure and Applied Mathematics*, vol. 37, 1905, p. 142, a Table of the values of $(p-1)\div\xi$ is given—for prime moduli (p)—up to $p>10000$; from which the value of ξ can be found by a trifling division.

$$\therefore\ P=\frac{2^{\xi}-1}{N}, \qquad \text{.....(8)}$$

i.e. P is the quotient of $(2^{\xi}-1)\div N$ (in the binary scale). .....(8*a*)

The quantity P, thus found, will be an integer, containing only $(\xi-\alpha)$ digits, beginning and ending with a unit. A group of α ciphers, with the binal point on the left, should be prefixed to form the binal fraction $P/2^{\xi}$, which is one period of $1/N$. If points be now placed over the first and last figures to show that the whole period is *recurrent*, this result will be the complete binal expression of $1/N$.

Hence, if N_1, N_2 be co-factors of $(2^{\xi}-1)$, and P_1, P_2 be the periods of $1/N_1$, $1/N_2$ respectively; then

$$P_1=\frac{2^{\xi}-1}{N_1}=N_2, \text{ and } P_2=\frac{2^{\xi}-1}{N_2}=N_1. \qquad \text{.....(9)}$$

Ex.
$$N=(2^{11}-1)=23\cdot 89=N_1\cdot N_2,$$
$$P_1(\text{of } 23)=\tfrac{1}{23}\cdot(2^{11}-1)=89=[1011001],$$
$$P_2(\text{of } 89)=\tfrac{1}{89}\cdot(2^{11}-1)=23=[10111].$$

Hence $\frac{1}{23}=\cdot\dot{0}000101101\dot{0}$; $\frac{1}{89}=\dot{0}0000101\dot{1}\dot{1}$.

It is now easy to find the *least* number N_2 such that the period P_2 of $1/N_2$ shall be a given number N_1 in binary scale preceded by ciphers. It suffices to find the Haupt-Exponent ξ of 2 to modulus N_1; then, by (9), taking

$$N_2=\frac{1}{N_1}(2^{\xi}-1), \qquad \text{.....(9}a\text{)}$$

it follows that $P_2(\text{of } 1/N_2)=N_1$, as required.

Similarly if $m\xi$ be any multiple of the above ξ, and

$$\mathbf{N}_2=\frac{1}{N_1}(2^{m\xi}-1) \quad [=\text{an integer obviously}], \qquad \text{.....(9}b\text{)}$$

then all numbers of type $\mathbf{N}_2$ have the period $P_2(\text{of } 1/\mathbf{N}_2)=N_1$ as required.

Although the Rule just given (9) is quite sufficient for computing the complete periods (P) of the Binal Fractions of any number $1/N$, it is thought that an investigation of the properties of these fractions may be interesting; this will be found to give in many cases simpler methods for finding $1/N$. In most cases it has been thought sufficient to merely state those properties, and to illustrate them by examples, without giving formal proofs.

5. *Simple cases* $[N=(2^{\alpha}\mp 1)]$. The periods (P) can be written down at sight.

$N=(2^{\alpha}-1)$; $P=\cdot\dot{0}00\ldots 0\dot{1}$ [$(\alpha-1)$ ciphers, 1 unit]. .....(10*a*)

$N=(2^{\alpha}+1)$; $P=\cdot\dot{0}00\ldots 011\ldots\dot{1}$ [α ciphers, α units]. .....(10*b*)

$N=(2^{\alpha+1}-1)$; $P=\cdot\dot{0}00\ldots 0\dot{1}$ [α ciphers, 1 unit]. .....(10*c*)

Hence the reciprocal $(1/N)$ of all numbers (N) from $N=(2^{\alpha}+1)$ to $(2^{\alpha+1}-1)$ have their periods (P) beginning with a group of α ciphers: the remaining $(\xi-\alpha)$ figures of the period contains a certain number (U) of units gradually decreasing from $U=\alpha$ when $N=(2^{\alpha}+1)$, as N increases, to $U=1$ when $N=2^{\alpha+1}-1$; but this group of $(\xi-\alpha)$ figures always begins and ends with a unit.

6. *Factors of* $(2^{r\alpha}-1)$.

When N_r is given by the succession-formula

$$N_{r+1}=2^{\alpha}\cdot N_r+1 \quad [\text{with } N_0=1], \qquad \text{.....(11)}$$

then $$N_r=\frac{2^{(r+1)\alpha}-1}{2^\alpha-1}, \quad \ldots\ldots(12)$$

or $$N_r=100\ldots100\ldots100\ldots01 \quad \ldots\ldots(13)$$

$[(r+1)$ units, $(\alpha-1)$ ciphers in sub-period, binary scale],

then $$P_r=\cdot\dot{0}00\ldots0111\ldots\dot{1} \quad [r\alpha \text{ ciphers, } \alpha \text{ units}]. \quad \ldots\ldots(14)$$

A few of these numbers (N_r) are shown in the Table below (in the denary scale), with the general expressions of N_r and P_r in binary scale.

α	N (denary). $r=0$,	1,	2,	3,	4	N (binary). $(r+1)$ units.	Period P. $r\alpha$ ciphers, α units.
1	1,	3,	7,	15,	31	111 1	$\cdot\dot{0}00\ldots\dot{1}$
2	1,	5,	21,	85,	341	10101 1	$\cdot\dot{0}00\ldots1\dot{1}$
3	1,	9,	73,	585,	4681	1001001 1	$\cdot\dot{0}00\ldots11\dot{1}$
4	1,	17,	273,	4369,	69905	100010001 ... 1	$\cdot\dot{0}00\ldots111\dot{1}$
5	1,	33,	1057,	—	—	100001 1	$\cdot\dot{0}00\ldots1111\dot{1}$

It will be seen that [see Art. 5]

The top line is $N=2^{r+1}-1$, and the second column is $N=2^\alpha+1$.

7. *End figures of P.*

Let $N=2^\alpha+B$, where $B<2^\alpha$.

Let P, Q be the periods of $1/N$ and $1/B$.

Then $$P\equiv Q \pmod{2^\alpha}, \quad \ldots\ldots(15)$$

so that

The end group of α figures of $1/N$, $(N>2^\alpha)$, is the same as the end group of α figures of $1/B=1/(N-2^\alpha)$. $\ldots\ldots(15\alpha)$

Hence $(2\alpha+1)$ figures of the periods (P) of $1/N$ are known for all numbers $N>2^\alpha$, viz. α ciphers and one unit at the beginning (Art. 3), and the end groups of α figures, as above (if that group be known for the period of $1/B$); this leaves only $(\xi-2\alpha-1)$ figures undetermined.

Hence also, the *complete period* (P) of $1/N$ may be written down at sight for all numbers $N>2^\alpha$, but $<2^{\alpha+1}$, such that $\xi=2\alpha+1$, if the end group of α figures of $1/B=1/(N-2^\alpha)$ be known.

But, if $\xi<(2\alpha+1)$, it suffices to determine the end groups of $(\xi-\alpha-1)$ figures; for

If $$N=2^\alpha+B\equiv B' \pmod{2^{\xi-\alpha-1}}$$

and P, Q' be the periods of $1/N$, $1/B'$,

then $$P\equiv Q' \pmod{2^{\xi-\alpha-1}}, \quad \ldots\ldots(16)$$

so that

The end groups of $(\xi-\alpha-1)$ figures of the periods of $1/N$ and $1/B'$ are the same. $\ldots\ldots(16\alpha)$

Hence, it suffices to prefix α ciphers and one unit (Art. 3) to the end groups of $(\xi-\alpha-1)$ figures found by (16α) to obtain the complete period of $1/N$.

Ex. Take

$N=$	73,	85,	89,	91,	95,	all $<2^7$ $(\alpha=6)$,
$n=\xi=$	9,	8,	11,	12,	10,	all $<2\alpha+1$,
$\xi-\alpha-1=$	2,	1,	4,	5,	3,	
$B'=$	1,	1,	9,	27,	7.	

Here the end groups of $(\xi-\alpha-1)$ figures are readily found from the periods of $1/B'$.

8. *Products of Fermat's Numbers.* The complete periods of the products of these numbers can often be readily found by the Rules of Art. 7.

Let $X=2^x$, $Y=2^y$, etc. ... $M=2^m$.
Let $F_x=2^X+1$, $F_y=2^Y+1$, etc., $F_m=2^M+1$.
Then F_x, F_y, etc., are Fermat's Numbers.
Let ξ_x, ξ_y, ... ξ_m be the Haupt-Exponents (mod F_x, F_y, etc.).

Then $$\xi_x=2^{x+1},\ \xi_y=2^{y+1},\ \ldots\ \xi_m=2^{m+1}. \quad \ldots\ldots(17)$$

Now let $x<y<\ldots<m$.
Then ξ_x, ξ_y, etc. ... are factors of ξ_m.
And, if $N=F_x \cdot F_y \ldots F_m$ (any product ending with F_m),

then $$\xi=\xi_m=2^{m+1}. \quad \ldots\ldots\ldots\ldots(18)$$

One of the most interesting series of products is

$$N=F_m \cdot F_{m-1} \cdot F_{m-2} \cdot \ldots \cdot F_{m-r}, \quad \ldots\ldots(19)$$

where N is a product of *successive* Fermat's Numbers from F_m down to F_{m-r}. Here $\xi=\xi_m=2^{m+1}$, and it is easy to see that the complete period (P) of $1/N$ consists simply of a group of c ciphers followed by a group of U units, where

$$c=a=2^m+2^{m-1}+2^{m-2}+\ldots+2^{m-r}=2^{m-r}.(2^{r+1}-1). \quad \ldots\ldots(19a)$$

$$U=\xi-c=2^{m-r}. \quad \ldots\ldots\ldots\ldots(19b)$$

The Table below shows some simple examples of the above: the values of c, U suffice to give the *complete* period (P) of $1/N$.

m, r	N	ξ	$c=a$	U
2, 1	$5 \cdot 17=85$	8	6	2
3, 1	$17 \cdot 257=4369$	16	12	4
4, 1	$257 \cdot 65537$	32	24	8
3, 2	$5 \cdot 17 \cdot 257=21845$	16	14	2
4, 2	$17 \cdot 257 \cdot 65537$	32	28	4
4, 3	$5 \cdot 17 \cdot 257 \cdot 65537$	32	30	2

8a. *Periods ending in* 100 ... 1. It is now easy to write down at sight the whole series of numbers N such that the period P of $1/N$ shall consist of ciphers followed by the figures 101, 10001, etc., expressing any Fermat's number (F_x) in the binary scale.

Take any number $m>x$, so that $F_x<F_{m-1}$.

Let D=product of all the Fermat's Numbers up to F_{m-1}, excluding F_x,

$$=F_0F_1F_2 \ldots F_{x-1} \cdot F_{x+1} \ldots F_{m-1} \quad \ldots\ldots(20)$$

$$=\frac{1}{F_x}(F_0F_1F_2 \ldots F_{m-1}, \text{ including } F_x)$$

$$=\frac{1}{F_x}(2^M-1), \text{ where } M=2^m; \quad \ldots\ldots(20a)$$

$$\therefore\ F_x=\frac{1}{D}(2^M-1). \quad \ldots\ldots(20b)$$

Then, by (9), all numbers of type D have the periods (P) of $1/D$ ending in the number F_x=100...01 (with 2^{x-1} ciphers), preceded only by $c=(2^m-2^x-1)$ ciphers.

Examples in the Table below: the values of c shown together with the end groups of figures make up the complete period (P) of the fractions $1/D$.

x	m	D	Fx	ξ	Period (P) of $1/D$. $c=\alpha$	Period (P) of $1/D$. End-group.
1	3	$3\cdot17=51$	5	8	5	101
1	4	$3\cdot17\cdot257$	5	16	13	101
1	5	$3\cdot17\cdot257\cdot65537$	5	32	29	101
2	4	$3\cdot5\cdot257$	17	16	11	10001
2	5	$3\cdot5\cdot257\cdot65537$	17	32	27	10001

9. ξ *even.* When ξ is even, the period P of $1/N$ consists—in certain cases—of two conjugate semi-periods (say P', P'') of $\frac{1}{2}\xi$ figures each, such that

$$P+P''=2^{\frac{1}{2}\xi}-1=111\ldots1 \quad [\tfrac{1}{2}\xi \text{ units}], \qquad \ldots\ldots(21)$$

whence also

$$C=U=\tfrac{1}{2}\xi. \qquad \ldots\ldots(21a)$$

The cases in question are [when ξ is *even*]:

1°. When N is prime. 2°. When N is a power of a prime. ……(22*a*)

3°. When $N=p_1p_2p_3\ldots p_r$, } where $p_1\equiv p_2\equiv p_3=\ldots3 \pmod 8$. ……} (22*b*)

4°. When $N=p_1^{\alpha}p_2^{\beta}p_3^{\gamma}\ldots p_r^{\rho}$, } or $p_1\equiv p_2\equiv p_3=\ldots5 \pmod 8$. ……} (22*c*)

5°. When $N=2^m+1$. ……(22*d*)

Examples will be found in the Table, at end; the semi-periods are marked by the symbol ∧ separating them.

Ex. of 2°; $N=9, 25, 27, 81$.

Ex. of 3°; $N=33, 57, 65$. *Ex.* of 4°; $N=99$.

Ex. of 5°; see Art 5 (the whole period is known).

Hence, with the notation of Art. 7,

P' begins with α ciphers followed by a unit. ……(23*a*)

P'' begins with α units followed by a cipher. ……(23*b*)

The end group of α figures of P'' is known from the period of $1/B$, and the end group of α figures of P' will be the complement thereof. ……(23*c*)

Hence $(4\alpha+2)$ figures of the period of $1/N$ are known for all numbers $N>2^{\alpha}$ of the kinds 1°, 2°, 3°, 4° above.

Hence also the *complete period* (P) of $1/N$ may be written down at sight for all numbers $N>2^{\alpha}$ but $<2^{\alpha+1}$ of the kinds 1°, 2°, 3°, 4° above, when $\frac{1}{2}\xi=2\alpha+1$, if the end group of α figures of $1/B=1/(N-2^{\alpha})$ be known.

But, if $\frac{1}{2}\xi<(2\alpha+1)$, it suffices to determine the end group of $(\frac{1}{2}\xi-\alpha-1)$ figures: for—as in Art. 7—

If $$N=2^{\alpha}+B\equiv B'' \pmod{2^{\frac{1}{2}\xi-\alpha-1}},$$

and, if P, Q'' be the periods of $1/N$, $1/B''$,

then $$P\equiv Q'' \pmod{2^{\frac{1}{2}\xi-\alpha-1}}, \qquad \ldots\ldots(24)$$

so that

The end groups of $(\frac{1}{2}\xi-\alpha-1)$ figures of the periods of $1/N$ and $1/B''$ are the same. ……(24*a*)

Hence, to form the second semi-period (P'') of $1/N$ (when N is one of the numbers 1°, 2°, 3°, 4° above), it suffices to prefix a group of α units and one cipher to the end group of $(\frac{1}{2}\xi - \alpha - 1)$ figures given by (24a).

And, for the first semi-period (P'), write down the leading group of α ciphers (after the binal point) and one unit, and follow with $(\frac{1}{2}\xi - \alpha - 1)$ figures formed as the complement of the figures given by (24a).

Ex. Take $N=41,\ 43,\ 57$, each $>2^5$ but $<2^6$.

$\frac{1}{2}\xi=$	10,	7,	9.
$\alpha=$	5,	5,	5.
$(\frac{1}{2}\xi-\alpha-1)=$	4,	1,	3.
$B''=$	9,	1,	1.

Here the end groups of 4, 1, 3 figures of P'' may be taken from the end groups of $1/B''=1/9,\ 1/1,\ 1/1$ respectively.

10. $N=(2^\alpha+2^{\alpha-1}\mp1)$. It might be inferred—from the last paragraph of Art. 5—that the leading 2α figures of the periods of the reciprocals $(1/N)$ of these two numbers would be closely alike. In fact it will be found that

If P_1 be the period of $1/N_1$, where $N_1=2^\alpha+2^{\alpha-1}-1$,

P_2 be the period of $1/N_2$, where $N_2=2^\alpha+2^{\alpha-1}+1$,

then each of P_1, P_2 have $c=\alpha$ leading ciphers, followed by $(\alpha-1)$ figures of 101010... ; *i.e.* $(2\alpha-1)$ figures alike. .....(25a)

And the next ($2\alpha^{\text{th}}$) figure is a 1 in P_1, and a 0 in P_2. .....(25b)

For examples, see $N=11,\ 13$; $23,\ 25$; $47,\ 49$; $95,\ 97$ in Table at end.

10a. $N=(2^\alpha\mp3)$. The periods (P) of $1/N$, where $N=(2^\alpha\mp3)$ might be expected to be in contrast, much in the same way as when $N=(2^\alpha\mp1)$ [Art. 5]. In fact it will be found that

If P_1 be the period of $1/N_1$, where $N_1=2^\alpha-3>5$,

P_2 be the period of $1/N_2$, where $N_2=2^\alpha+3>7$,

then

P_1 begins with $c=(\alpha-1)$ ciphers, then a 1 ; then $(\alpha-2)$ ciphers, then 11, etc. (26a)

P_2 begins with $c=\alpha$ ciphers, then $(\alpha-2)$ units, then 01, etc. ....(26b)

For examples, see $N=13,\ 19$; $29,\ 35$; $61,\ 67$ in Table at end.

11. *Table of Binal Fractions* $(1/N)$. The Table at end of this Paper gives the following data, for all *odd* numbers $N\not>100$, to Argument N.

Col. 1. N in denary scale. Col. 2. N in binary scale.

Col. 3. The number of figures (n) in the Repetend Period (P) of the Binal Fraction $1/N$.

Col. 4. The Repetend Period (P) of the Binal Fraction $1/N$ properly pointed so as to show the repetend (Art. 1—1°, 3°).

When the repetend period (P) consists of two conjugate semi-periods (P', P'')—see Art. 9—a circumflex (^) has been placed between them to mark the point of separation.

[The repetends (P) are printed for clearness in groups of four figures slightly separated, and the numbering of the digits in P is printed (in small type) in the heading to facilitate the counting the number of digits.]

TABLE OF BINAL FRACTIONS ($1/N$).

[n=number of figures in Repetend (P) of $1/N$.]

N	N (Binary)	n	Repetend (P) of Binal Fraction ($1/N$). 4 8 12 16 20 24 28 32 36 40 44 48 52 56 60 64 68 72 76 80 82
1	1	1	·i
3	11	2	·oi
5	101	4	·o^oi
7	111	3	·o^i
9	1001	6	·o^oi ii
11	1011	10	·o^oi o^iii oi
13	1101	12	·o^oi o^oii ioi
15	1111	4	·o^oi
17	1 0001	8	·o^ooo iiii
19	1 0011	18	·o^ooo iioi oiii ioo oi
21	1 0101	6	·o^ooo ii
23	1 0111	11	·o^ooo ioii ooi
25	1 1001	20	·o^ooo ioio ooii iioi oiii
27	1 1011	18	·o^ooo ioo oiii ioii oi
29	1 1101	28	·o^ooo iooo iioi ooii iioi iioo ioii
31	1 1111	5	·o^ooo i
33	10 0001	10	·o^ooo oiii ii
35	10 0011	12	·o^ooo oiii oioi
37	10 0101	36	·o^ooo oiio iiio ioii ooii iiio oioo oioi ooii
39	10 0111	12	·o^ooo oiio ioo i
41	10 1001	20	·o^ooo oiio ooii iiio oiii
43	10 1011	14	·o^ooo oioi iiii oi
45	10 1101	12	·o^ooo oioi ioii

45	10 1101	12	·0000 0101 1011
47	10 1111	23	·0000 0101 0111 0010 0110 001
49	11 0001	21	·0000 0101 0011 1001 0111 1
51	11 0011	8	·0000 0101
53	11 0101	52	·0000 0100 1101 0100 1000 0111 0011 1110 1100 1010 1101 1110 0011
55	11 0111	20	·0000 0100 1010 0111 1001
57	11 1001	18	·0000 0100 0111 1101 11
59	11 1011	58	·0000 0100 0101 0110 1100 0111 1001 0111 1101 1101 0100 1001 1100 0011 01
61	11 1101	60	·0000 0100 0011 0010 0101 1100 0101 0011 1110 1111 0011 0110 1000 1110 1011
63	11 1111	6	·0000 01
65	100 0001	12	·0000 0011 1111
67	100 0011	66	·0000 0011 1101 0010 0010 0110 0011 0101 0111 1110 0001 0110 1110 1100 1110 0101 01
69	100 0101	22	·0000 0011 1011 0101 1100 11
71	100 0111	35	·0000 0011 1001 1011 0000 1010 1101 0001 001
73	100 1001	9	·0000 0011 1
75	100 1011	20	·0000 0011 0110 1001 1101
77	100 1101	30	·0000 0011 0101 0011 0001 1101 1110 11
79	100 1111	39	·0000 0011 0011 1101 1001 0001 1101 0010 1010 001
81	101 0001	54	·0000 0011 0010 1001 0001 0110 0001 1111 1001 1010 1101 1101 0011 11
83	101 0011	82	·0000 0011 0001 0101 1001 0111 0010 0001 1110 1101 0111 1110 0111 0101 0011 0100 0110 1111 0000 1001 01
85	101 0101	8	·0000 0011
87	101 0111	28	·0000 0010 1111 0001 0100 1001 1001
89	101 1001	11	·0000 0010 111
91	101 1011	12	·0000 0010 1101
93	101 1101	10	·0000 0010 11
95	101 1111	36	·0000 0010 1011 0001 1101 1010 0100 0110 0001
97	110 0001	48	·0000 0010 1010 0011 1010 0000 1111 1101 0101 1100 0101 1111
99	110 0011	30	·0000 0010 1001 0101 1111 1010 1101 01.

REVIEWS.

Clive's Mathematical Tables. (University Tutorial Series.) **Four Figure Tables and Constants.** By W. HALL. (Cambridge University Press.)

In the first of these a claim is made in the preface that the tables given have been specially designed to combine speed with accuracy of calculation, and on examination they are found to justify it. Each separate table is printed on two facing pages, and a high degree of accuracy is obtained by the following methods:

(1) The tabulated functions are given to 5 significant figures.

(2) In places where the mean differences change too rapidly for them to be given for a whole line of entries they are given for each half line, and the blank spaces of the entry columns are utilised by the insertion of a few figures indicating the amount of error in the mean difference given.

(3) In places where no reliable mean differences can be given the degree of accuracy to be obtained by ordinary interpolation is indicated.

Thus the cosecant of 49° 27′ is found by adding to the entry for 49° 24′ (1·53663) the 'tabular difference' 155 and the 'correction' 2 obtained by multiplying the small correction supplied for 1 mi. just below the entry for 49° 24′ by the number of extra minutes for which the difference is required. We thus obtain 1·53820, which is the value given in Lodge's Appendix to Bremiker's *Logarithms*. We have checked various other entries by Bremiker. Careful instructions with fully worked examples are given, and sufficient theory for the students to understand the principles on which logarithm work is based.

Mr. Hall's work is intended for nautical as well as for ordinary technical and physical calculations, and therefore contains some tables which are not to be found in ordinary sets. For instance, the 'Haversine' (half versed sine) and its logarithm are given to 4 places at intervals of 6′, and examples are given of the use of this function in calculations, especially in those of Spherical Trigonometry,

$$e.g.\ \text{since}\ \operatorname{Hav}\theta=\tfrac{1}{2}(1-\cos\theta)=\sin^2\frac{\theta}{2},$$

$$\therefore\ \text{in Pl. Trig.}\ \operatorname{Hav}A=\frac{(s-b)s-c)}{bc};$$

and in Spherical Trigonometry

$$\begin{aligned}\text{since}\quad \cos a&=\cos b\cos c+\sin b\sin c\cos A\\&=\cos(b-c)-\sin b\sin c\operatorname{hav}A,\\ \operatorname{hav}a&=\operatorname{hav}(b-c)+\operatorname{hav}\theta,\\ \text{where}\quad \operatorname{hav}\theta&=\sin b\sin c\operatorname{hav}A.\end{aligned}$$

It may interest those readers of the *Gazette* who do not happen to have worked with the Haversine to work out the solution of a problem so as to be able to compare the result obtained by an ingenious graphic method to be mentioned later.

To find a, when $b=75°$, $c=55°$, $A=70°$.

$\log\operatorname{hav}70°=\bar{1}\cdot5172$	$\operatorname{hav}\theta=\cdot2597$
$\log\sin75°=\bar{1}\cdot9849$	$\operatorname{hav}20°=\cdot0302$
$\log\sin55°=\bar{1}\cdot9134$	$\operatorname{hav}A=\cdot2899$
$\log\operatorname{hav}\theta=\bar{1}\cdot4155$	$A=65°\ 9'$

A traverse table (a storehouse of easy examples for school use) in right-angled triangles, besides a table of chords in addition to those of functions usually given, add to the usefulness of the work. It is moreover beautifully printed in several carefully graduated sizes of type.

Principes et Formules de Trigonométrie Rectiligne et Sphérique. Par J. PIONCHON. (Gauthier Villars.) 5 francs.

This serviceable work belongs to the Bibliothèque de l'Elève-Ingénieur. It contains all the formulae an engineering student is likely to want in the subjects of which it treats, with sufficient theory to shew how they are derived, and includes the simpler cases of definite and indefinite integration. The treatment

of the $\int_0^{\pi} \sin^2 x dx$ is particularly neat. The most interesting paragraph in the work we think to be that on the *Méthode graphique de résolution des triangles sphériques.*

Writing the formula $\cos a = \cos b \cos c + \sin b \sin c \cos A$,

$$2\cos a - (1+\cos A)\cos(b-c) - (1-\cos A)\cos(b+c) = 0,$$

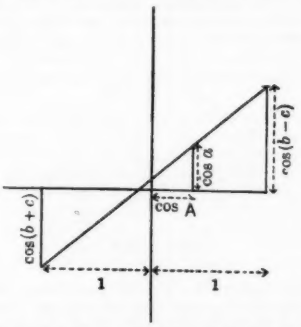
the author points out, after Collignon, that this gives

$$\begin{vmatrix} -1, & \cos(b+c), & 1 \\ 1, & \cos(b-c), & 1 \\ \cos A, & \cos a, & 1 \end{vmatrix} = 0,$$

and hence that the three points $(1, \cos\overline{b-c})$, $(-1, \cos\overline{b+c})$, $(\cos A, \cos a)$ are collinear. Hence marking the points $(1, \cos\overline{b-c})$, $(-1, \cos\overline{b+c})$ and joining them by a straight line, $\cos a$ is the ordinate of the point on it whose abscissa is $\cos A$. If, as the author suggests, we use d'Ocagne's 'abacus,'* constructed by ruling parallels to the coordinate axes through the points of division of these axes into scales of cosines, there is no need to consult tables. Recourse to tables may also be avoided by using Granville's *Plotting Circles for Polar Coordinates* to obtain the requisite cosines graphically and read off the final result. But as the only functions required are natural cosines, these can all be obtained from one opening in a book of 4-figure tables, and we find the method very expeditious on a sheet of Pye's paper ruled in millimetres, the white margin round the 20-centimetre square being distinctly convenient. The final reading for the angle A found by calculation above to illustrate the haversine formula to be 65° 9′ coming out 65° 30′. The method deserves a place by the side of the graphic method derived from R. F. Davis's elegant investigation *in plano* of the same formula (see *Mathematical Gazette*, vol. ii., p. 261). Although derived from the vanishing of a determinant, it seems more simply proved by pointing out that

$$y = \cos b \cos c + x \sin b \sin c$$

is satisfied by $(1, \cos\overline{b-c})$, $(-1, \cos\overline{b+c})$, and that if $x = \cos A$, $y = \cos a$.



Briefwechsel zwischen C. G. J. Jacobi und M. H. Jacobi. Herausgegeben von W. AHRENS.

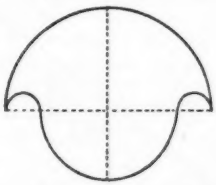
This part (the 22nd) of the series of 'Abhandlungen zur Geschichte der Mathematischen Wissenschaften' originated by Cantor is of great interest. To mathematicians the name of the younger brother, *Carl Gustav Jacob* (1804-1851),

* D'Ocagne's abacus is of course simply the rectangular network that would be constructed as in Daniell's *Text Book of the Principles of Physics*, pp. 80, 83, 84, for describing a series of ellipse traced by a point having two single harmonic motions at right angles to each other. In graduating the scales of cosines the origin is marked 90° in both cases, the number from 0° to 180° proceeding in the negative direction along each axis.

is of course the more familiar. That of the elder, Moritz Hermann (1801-1875), occupies however an honoured place in the history of electrical discovery, and just now may have a revived attention given to it as that of the inventor of the first 'motor boat.' As the two brothers corresponded freely with each other about their occupations, and as from their influential positions they had a wide acquaintance, including practically all the 'scientific worthies' of Europe, to whose work they continually allude, the correspondence forms a mine of information for the future historian, especially as the editor has added a large number of explanatory notes with reference to other publications. One of the most interesting to an English reader is that dated Sept. 25, 1842, written by the mathematician, in which he describes his visit to England with Bessel to attend officially the Manchester meeting of the British Association. "In Manchester sprach Faraday viel mit mir von Dir. Er reiste ab als die eigentliche Versammlung anfing." We were puzzled to account for Faraday's journey to Manchester when he obviously did not intend to stay for the general meeting. Bence Jones does not allude to the matter, but the Schönbein-Faraday correspondence makes it clear. Faraday had not intended to go to Manchester at all, but the Society of Sciences at Modena had appointed Herschel and him to represent them, and as the former had at first said that he could not go it fell to Faraday to attend a committee meeting and report the credentials of the Society. Herschel seems to have gone after all, and Jacobi's letter supplies the probable reason for his change of mind, viz. to meet Bessel, "if it were but to touch the garment of this gentleman."* An elaborate index of the names of the writers and subjects mentioned in the correspondence and notes adds greatly to the value of the work. Portraits of the two brothers are given.

Breve storia della matematica dai tempsi antichi al medio-evo. By Dr. G. Fazzari. (Landron.)

A very readable treatise, extending as its name implies from the earliest times to those of Leonardo da Vinci and Albert Dürer. Many specimens are given of the discoveries and style of work of the authors of various ages, with sufficient detail to make them easy to follow. They are frequently just such as might be used profitably in class work by teachers who wish to shew their pupils some of the theorems discovered by ancient geometers, or some of the curious methods of compilation used by ancient calculators. Of the geometrical theorems we may notice those connected with the ὄρβηλος and the σάλινον. The figure of the latter,



formed of 4 semi-circles as shewn, we had not seen before, and we do not find any allusion to it in Dr. Mackay's paper on the arbelos, *Proc. Edin. Math. Soc.*, Nov., 1884. The area of the figure bounded by the four semi-circles is equal to that of the circle on its axis of symmetry as diameter. The work seems to have been founded to some extent on that of Cajori, though we do not find any mention of the obligations of the author to the latter's history. Such striking coincidences as the following can scarcely be undesigned:

Cajori. "The first people who distinguished themselves in mathematical research after the time of the ancient Greeks belonged like them to the Aryan race. It was however not a European but an Asiatic nation, and had its seat in far-off India. Unlike the Greek, Indian society was fixed into castes. The only castes enjoying the privilege and leisure for advanced study and thinking were

* From the presidential address Herschel and Bessel seem to have met before reaching Manchester and to have travelled there together in the same train.

the Brahmins, whose prime business was religion and philosophy, and the Kshatriyas, who attended to war and government."

Fazzari. "Mentre la matematica declinava presso il popolo ellenico era coltivata da un altro popolo appartenente anch' esso alla razza degli Arii, abitante la regione bagnata dal Gange e dall' Indo. Ma quanta differenza fra' due populi! All' opposto dei Greci, gl' Indiani erano divisi in caste e fra queste era riservato il privilegio di dedicarsi agli stude solamente ai Bramani che si occupavano di religione e di filosofia ed ai Sciatria che attendavano alla guerra ed al governo.

Analytische Geometrie der Kegelschnitte nach Georg Salmon. By Dr. W. FIEDLER. (Teubner.)

There is we believe little difference between this (7th) edition and the 6th issued in 1898. But Dr. Fiedler has prefixed to this an appreciation of Salmon and an account of his work. In touching words he tells us that in his old age he counts his intimacy with Salmon as one of the most precious of the good things which have come to him in his course through life. He gives a short sketch of his work both as a mathematician and a theologian, and bears testimony to his devotion to duty, his kindly humour, his simple Christian faith, and his free but unostentatious benevolence. If anyone following the example of the Cambridge man instanced by Dr. Fiedler who chose Salmon's *Conics* as one of three books to be allowed him in banishment to a desert island, this edition can be recommended not only on account of this excellent sketch of Salmon, but for the copious historical notes supplied by the editor, though these indeed might by their interesting references to the works of Plücker, Steiner, Möbius, etc., make him long for access to a good mathematical library.*

Das Rechnen mit Vorteil. By F. ROGEL. (Teubner.) 80 pf.

We have here a collection of methods of arithmetical work specially designed for speed and accuracy. It may be recommended to those who make a specialty of computation work, for though some of the devices are fairly well known, there are others which are not found in ordinary text-books, such as the use of the 'arithmetical complement' in division. The method of turning a given fraction into the sum of a series of fractions such that each is obtained from the one preceding it by dividing it by an integer may also be mentioned.

Anschauliche Grundlagen der mathematischen Erdkunde. By Dr. KURT GEISSLER. (Teubner.)

An excellent treatise on elementary astronomy and mathematical geography. Starting from the description of simple observations on the form of the earth's surface and the motion of the stars, the student is led carefully on to the laws of Kepler and Newton. Clear diagrams are given and the apparatus described is simple and probably effective for its purpose; but 'earth-knowledge' is a subject which does not seem to get much time allowed to it in schemes of work or to be much valued by examining boards. The mathematical part probably falls 'between two stools,' though it is beginning to find its way into courses of 'Solid.' The book is to be highly recommended to teachers either of geography or of mathematics.

E. M. LANGLEY.

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*The mention of the use of mathematics in alleviating the tedium of banishment may well recall the investigations on projective methods carried on by Poncelet in his Russian prison. Perhaps the following instance is less known: "Newton (*i.e.* John Newton, afterwards the friend of Cowper) became the overseer of one of those depôts of slaves maintained at the mouths of the great African rivers. . . . But he sank into a bondage only less deplorable than that of his unhappy captives. . . . As he traversed the shore from one pestilential estuary to another, the unhappy outcast would have been as destitute of solace from within as from without had it not happened that a copy of Barrow's *Euclid* had stuck by him in all his wanderings, and as he traced the diagrams on the sand and revolved the demonstrations his sorrows took a temporary flight."—*Essays by Sir J. Stephen*, 1850.

ANSWERS TO QUERIES.

[32, p. 166]. The formula

$$1+nC_1^2+nC_2^2+\ldots=\frac{2n!}{n!\,n!}$$

is well known: Cf. Chrystal, *Algebra*, Vol. II. p. 18. When n is not restricted to positive integral values it is easy to deduce from the well-known formula

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},$$

$$1+\binom{n}{1}^2+\binom{n}{2}^2+\ldots=\frac{\Gamma(2n+1)}{\Gamma(n+1)\Gamma(n+1)}.$$

The sum for the cubes of the coefficients

$$1+\binom{n}{1}^3+\binom{n}{2}^3+\ldots=\frac{1}{2}\frac{\sin n\pi}{\sin\frac{n\pi}{2}}\frac{\Gamma\left(1-\frac{3n}{2}\right)}{\Gamma^3\left(1-\frac{n}{2}\right)}$$

was first given by Prof. Frank Morley. A paper by him dealing with the general case (n-non integral) will be found in *Proc. L.M.S.*, Vol. 34, p. 397. This paper contains references to an earlier paper: other series inspired by Prof. Morley's paper will be found in *Proc. L.M.S.*, Vol. 35, p. 284, due to Prof. A. C. Dixon. A short note on the sum of the q-form cubes by the present writer will be found *Proc. L.M.S.*, Series II., Vol. 3, notes page xxi.

F. H. Jackson.

THE INDIAN MATHEMATICAL CLUB.

We are pleased to report that the above society is making progress, the twenty members who joined at its formation in May last having all but trebled. This is eminently satisfactory, especially when we consider that the subscription is £1. 13s. 4d. per annum. One-third of the funds goes in working expenses, one-third in the purchase of mathematical books, and the rest in subscriptions to mathematical periodicals. "Progress Reports" are issued monthly, giving information as to the additions to the Library, journals received, etc. It will be remembered that among the objects of the Club is the circulation of books and publications among members scattered over a vast peninsula of 1½ million sq. miles. The subscription is lowered in special cases—poor and deserving students—and any presents to the new Library would, we feel sure, be gratefully received by the Librarian, Prof. R. P. Paranjpye, Fergusson College, Poona. We notice Mr. V. Ramaswami Aiyar has presented his copy of Vol. III. of the *Gazette*. The Editor will be glad to send to the Librarian a copy of Vols. I. and II., if the following numbers can be obtained from some member or members of the Association who happen to have duplicates, or who do not wish to bind and keep complete sets. The good work the youthful club is doing is an adequate excuse for this appeal. The numbers in question are: 7-9, 26-27, 31-33, 37, 38, 40-42.

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